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The upper triangular solutions to the three-state constant quantum Yang–Baxter equation

Jarmo Hietarinta
Department of Physics, University of Turku, 20500 Turku, Finland

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Abstract. In this article we present all non-singular upper triangular solutions to the constant quantum Yang–Baxter equation $R_{j_1j_2}^{k_1k_2}R_{k_1j_3}^{l_1k_3}R_{k_2k_3}^{l_2l_3} = R_{j_2j_3}^{k_2k_3}R_{j_1k_3}^{k_1l_3}R_{k_1k_2}^{l_1l_2}$ in the three-state case. The upper triangular ansatz implies 729 equations for 45 variables. Fortunately, many of the equations turned out to be simple, allowing us to start breaking the problem into smaller ones. In the end we had a total of 552 solutions, but many of them were either inherited from two-state solutions or subcases of others. The final list contains 35 non-trivial solutions, most of them new.

1. Introduction

In this paper we continue our work on systematically solving the constant quantum Yang–Baxter equation (YBE)

$$R_{j_1j_2}^{k_1k_2}R_{k_1j_3}^{l_1k_3}R_{k_2k_3}^{l_2l_3} = R_{j_2j_3}^{k_2k_3}R_{j_1k_3}^{k_1l_3}R_{k_1k_2}^{l_1l_2}. \tag{1}$$

In our previous paper [1] (for more details, see [2]), we solved the two-state problem completely, which involved 64 equations in 16 variables.

In general, (1) contains N^6 equations for N^4 unknowns, so in the present case of $N = 3$ we have 729 equations for 81 variables. At the moment this is too complicated for an exhaustive study and one must proceed step by step with suitable ansatz. The present ansatz is based on the two-state results: it turned out [1, 3, 4] that all two-state non-singular solutions were either

- Upper triangular: $R_{ij}^{kl} = 0$, if $j > l$ or $j = l$, $i > k$, or
- Even weight: $R_{ij}^{kl} = 0$, if $k + l \neq i + j \pmod 2$.

(In thinking of R_{ij}^{kl} as an $N^2 \times N^2$ matrix, we use the convention that the right-hand indices define the block.) Thus upper triangularity turned out to be a good ansatz for finding two-state solutions, and we hope it is equally productive when the number of states is three.

Of course, several three-state solution have been obtained over the years. For example in [5] the spectral parameter dependent solutions associated with Kac–Moody algebras were obtained. In special limits these solutions may reduce to upper triangular form.

2. Simple solutions

Let us first of all note that the YBE does have some 'easy' classes of solutions. That is, the solutions are easy to construct from previous solutions or for some other reason. (This does not mean that the other related structures (e.g. algebras) are necessarily less interesting.) In this section we will describe two such classes; the solutions belonging to them are not mentioned again when we discuss the results of our search.

2.1. Solutions inherited from lower number of states

A solution to (1) with a lower number of states can always be dressed to become a higher-state solution. There are at least two ways to do this.

2.1.1. Diagonal dressing Let $\tilde{R}$ be an M-state solution of (1) and $N > M$. Let μ be a selection of M numbers from $\{1,\ldots,N\}$, and define the N-state R-matrix as

$$R_{ij}^{kl} = \begin{cases} \tilde{R}_{ij}^{kl} & \text{when } i,j,k,l \in \mu \\ s_{ij}\delta_i^k\delta_j^l & \text{otherwise.} \end{cases} \tag{2}$$

When this is substituted into (1) we find further conditions on s as follows:

$$\begin{aligned} \tilde{R}_{ij}^{kl}\,(s_{mi}s_{mj} - s_{mk}s_{ml}) &= 0 \\ \tilde{R}_{ij}^{kl}\,(s_{im}s_{ml} - s_{km}s_{mj}) &= 0 \\ \tilde{R}_{ij}^{kl}\,(s_{im}s_{jm} - s_{km}s_{lm}) &= 0 \end{aligned} \tag{3}$$

$\forall i,j,k,l \in \mu$, $m \notin \mu$, no sum. (It is interesting to note that the conditions are exactly like those obtained in a different context in [6].) A trivial solution to (3) is given by $s_{ij} = 1$, but there can also be others (depending on the form of $\tilde{R}$).

In the present case we have $M = 2$ and $N = 3$ and there is only one index outside μ. As an example let us take $\mu = \{1,3\}$ and

$$\tilde{R} = \left(\begin{array}{cc|cc} 1 & . & . & . \\ . & p & 1-pq & . \\ \hline . & . & q & . \\ . & . & . & 1 \end{array}\right) \tag{4}$$

(for matrices we always write '.' in place of a '0', for better readability) and then the only remaining condition from (3) is $s_{12}s_{21} = s_{23}s_{32}$. Thus we can extend (4) to a three-sate solution with the four additional parameters a, b, x, y:

$$R = \left(\begin{array}{ccc|ccc|ccc} 1 & . & . & . & . & . & . & . & . \\ . & a & . & . & . & . & . & . & . \\ . & . & p & . & . & . & 1-pq & . & . \\ \hline . & . & . & by & . & . & . & . & . \\ . & . & . & . & x & . & . & . & . \\ . & . & . & . & . & b & . & . & . \\ \hline . & . & . & . & . & . & q & . & . \\ . & . & . & . & . & . & . & ay & . \\ . & . & . & . & . & . & . & . & 1 \end{array}\right). \tag{5}$$

2.1.2. Block dressing The starting point is as above but the new higher-sate solution is constructed as follows:

$$R_{ij}^{kl} = \begin{cases} \tilde{R}_{ij}^{kl} & \text{when } i,j,k,l \in \mu, \\ \delta_i^k F_j^l & \text{when } j,l \in \mu,\ i,k \notin \mu \\ G_i^k \delta_j^l & \text{when } i,k \in \mu,\ j,l \notin \mu \\ \delta_i^k \delta_j^l & \text{otherwise.} \end{cases} \tag{6}$$

This ansatz leads to the conditions

$$\begin{aligned}
&(F\otimes F)\tilde{R}=\tilde{R}(F\otimes F)\\
&(1\otimes F)\tilde{R}(G\otimes 1)=(G\otimes 1)\tilde{R}(1\otimes F)\\
&\tilde{R}(G\otimes G)=(G\otimes G)\tilde{R}\\
&[F,G]=0.
\end{aligned}\tag{7}$$

The most important class of solutions to (7) are those with $G\propto F^{-1}$, in which case the first equation of (7) is sufficient. An example of such a solution is

$$R:=\left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & 1 & \cdot & \cdot & \cdot & -1 & \cdot & c_9\\
\cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & d_6 & \cdot\\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & c_9\\
\hline
\cdot & \cdot & \cdot & g_6 & \cdot & d_6g_6 & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & f_5 & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & g_6 & \cdot & \cdot & \cdot\\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -c_9\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right)\tag{8}$$

which decomposes as

$$\tilde{R}:=\left(\begin{array}{cc|cc}
1 & 1 & -1 & c_9\\
\cdot & 1 & \cdot & c_9\\
\hline
\cdot & \cdot & 1 & -c_9\\
\cdot & \cdot & \cdot & 1
\end{array}\right)\qquad F:=\begin{pmatrix}-1 & d_6\\ \cdot & -1\end{pmatrix}\qquad G:=\begin{pmatrix}g_6 & d_6g_6\\ \cdot & g_6\end{pmatrix}.\tag{9}$$

2.2. *Solutions generated by commuting matrices*

The following solutions can be said to be inherited from lower dimension, when dimension means the number of pairs of indices. (This hierarchial structure is more obvious when one considers extensions, see e.g. [7].)

The result is simply the following: Let $\{N(\alpha),M(\alpha)|\alpha\in I\}$ be a set of commuting $N\times N$ matrices, then it is easy to show that

$$R^{kl}_{ij}=\sum_{\alpha\in I}N(\alpha)^k_iM(\alpha)^l_j\tag{10}$$

is an N-sate solution of (1).

A commuting set of matrices can be simultaneously brought to the Jordan canonical form, but the Jordan form does not have to be diagonal. A two-state example is provided by

$$R=\left(\begin{array}{cc|cc}
1 & a & b & c\\
\cdot & 1 & \cdot & b\\
\hline
\cdot & \cdot & 1 & a\\
\cdot & \cdot & \cdot & 1
\end{array}\right).\tag{11}$$

It can be decomposed (this is not unique) as

$$N(1)=\begin{pmatrix}1 & a\\ \cdot & 1\end{pmatrix}\ M(1)=\begin{pmatrix}1 & \cdot\\ \cdot & 1\end{pmatrix}\ N(2)=\begin{pmatrix}b & c\\ \cdot & b\end{pmatrix}\ M(2)=\begin{pmatrix}\cdot & 1\\ \cdot & \cdot\end{pmatrix}.\tag{12}$$

When the number of states is three there is still more freedom and there are more varied solutions of this type, but we will not include such solutions in the list of results. An example is

$$R := \left(\begin{array}{ccc|ccc|ccc} 1 & . & 1 & . & a_5 & a_6 & c_9 & a_8 & a_9 \\ . & 1 & . & . & . & d_8a_5 & . & c_9 & d_8a_8 \\ . & . & 1 & . & . & . & . & . & c_9 \\ \hline . & . & . & 1 & . & 1 & . & d_8a_5 & d_8a_6 \\ . & . & . & . & 1 & . & . & . & d_8^2a_5 \\ . & . & . & . & . & 1 & . & . & . \\ \hline . & . & . & . & . & . & 1 & . & 1 \\ . & . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & . & 1 \end{array}\right).$$

3. Symmetries

It is well know that the set of equations (1) is invariant under the continuous transformation

$$R \to \kappa(Q \otimes Q)R(Q \otimes Q)^{-1} \tag{13}$$

where Q is any non-singular $N \times N$ matrix and κ a non-zero number. They are also invariant under the index reflections

$$R^{kl}_{ij} \to R^{ij}_{kl} \tag{14}$$

$$R^{kl}_{ij} \to R^{lk}_{ji} \tag{15}$$

the first of these is the usual matrix transposition and the second follows from $R \to PRP$, where $P^{kl}_{ij} = \delta^l_i \delta^k_j$ is the permutation operator.

For special choices of Q one can obtain other discrete transformations. In general, if σ is a permutation of the set $\{1, \ldots, N\}$, then $Q^i_j = \delta^i_{\sigma(j)}$ yields a relabeling of indices $R^{kl}_{ij} \to R^{\sigma(k)\sigma(l)}_{\sigma(i)\sigma(j)}$. An important special case of this is given by $Q^i_j = \delta^i_{N-j+1}$ which, when followed by transposition, yields reflection across the secondary diagonal.

In this paper we restrict the R-matrix to be an upper triangular matrix, after which it is natural to use only upper triangular transformation matrices Q, which we will call U from now on. Under such tranformations the upper triangular nature of R is preserved. Furthermore, one easily finds that the diagonal blocks transform as

$$R^{km}_{im} \to U^{i'}_i R^{k'm}_{i'm} U^{-1}{}^k_{k'} \qquad \text{(no sum over } m). \tag{16}$$

Thus, in order to fix the remaining rotational freedom we will just need to impose conditions (of a 'canonical form') on the diagonal blocks.

Normally one would require the canonical form of an upper triangular matrix be the Jordan canonical form but this may require transformation matrices that are not upper triangular. For this reason we must relax the definition of what is a canonical form. It turns out that using just upper triangular transformation matrices one can bring any upper triangular matrices into one of the following five 'semi-canonical' forms

$$C_1 = \begin{pmatrix} a & . & . \\ . & b & . \\ . & . & c \end{pmatrix} \qquad C_2 = \begin{pmatrix} a & b & . \\ . & a & . \\ . & . & c \end{pmatrix} \qquad C_3 = \begin{pmatrix} a & . & b \\ . & c & . \\ . & . & a \end{pmatrix}$$

$$C_4 = \begin{pmatrix} a & . & . \\ . & b & c \\ . & . & b \end{pmatrix} \qquad C_5 = \begin{pmatrix} a & b & . \\ . & a & b \\ . & . & a \end{pmatrix}. \tag{17}$$

This is the basis of our classification scheme.

4. How the equations were solved

4.1. Breakdown into smaller sets

As usual, we must use the available symmetries to divide the problem into several smaller ones. In particular, we want to fix the continuous symmetries related to upper triangular transformation matrices. The detailed breakdown is given in the appendix, here we give just the general idea.

We start with the top diagonal block and transform it into the semi-canonical Jordan form (17). We may therefore assume that the upper block is of the type C_i in (17). We work through the cases in opposite order, C_5 first. Using a reflection across the antidiagonal when necessary we may assume that the lowest diagonal block cannot be reflected up and transformed into anything that has already been analysed. Using the notation that $\sim C_n$ is anything that can be transformed to C_n using upper triangular matrices, we can write the first division by giving the upper and lower blocks:

(i) up C_5, down anything
(ii) up C_4, down anything but $\sim C_5$,
(iii) up C_3, down $\sim C_3$, or $\sim C_4$, or $\sim C_1$,
(iv) up C_2, down $\sim C_4$, or $\sim C_1$,
(v) up C_1, down $\sim C_1$.

In the third and fourth cases note that an antidiagonal reflection takes C_4 into C_2.

This the starting point. Further complications arise when $c = a$ in C_3 or $a = b = c$ in C_1, because then they commute with an upper triangular transformation matrix having units on the diagonal. In those cases one can use the remaining freedom to operate on the other diagonal blocks. In the appendix we have given a more detailed breakdown.

4.2. Solving the equations

For each subcase we still have to solve the 729 equations. Some of these equations vanish automatically because of the upper triangular ansatz. It turns out that there are also many simple equations that factor to linear factors. Since the present equation solver routines cannot handle well very large sets of equations, we decided to start the solution process by interactively splitting the process into branches and subbranches based on simple factorizable equations. (All algebraic work was done using REDUCE [8].) The algorithm was as follows:

(i) Initialize, in particular record the expression that cannot vanish (Ex: $A \neq 0$).

(ii) Are there simple factorizable equations?

Yes: Choose a simple equation (Ex. $B(A-1)(C-D)=0$) and find its solutions. Go to (iii).

No: Go to (iv)

(iii) Are there any solutions that do not break the non-vanishing condition?

Yes: Add the first solution as an additional rule to the current branch and create a new branch (which inherits the current assigments and non-vanishing condition) for each additonal allowed branch, if any. With each later branch add the condition that the assignment of the previous branch cannot hold. Go to (ii). (Example: $\{\{\{B => 0\}, A\}, \{\{A => 1\}, AB\}, \{\{C => D\}, AB(A-1)\}\}$)

No: Go to (v)

(iv) Are there equations you want to solve by hand?

Yes: Do it, but if you have to assume something, create also a branch where the assumption does not hold. Go to (iii).

No: Solve the remaining equations using 'groesolve' [9] and output the allowed solutions. Go to (v).

(v) Is this the last branch?

Yes: End.

No: Take the next branch. Go to (ii),

A log of the interactive solution process was saved using the unix 'tee' utility, and checked later. Altogether the printout extended well over 1000 pages. In the end we had 552 solutions, but naturally most of them turned out to be of the simple type of section 2, or subcases of other solutions.

5. The non-trivial solutions

As expected, the problem of classifying the solutions is almost as time-consuming as finding them. We must characterize the solutions by something that is invariant under the possible transformations. The method we finally chose was to classify first by the number of different eigenvalues of the R-matrix. In counting the different eigenvalues one must be careful, because some solutions only seem to have the required number of eigenvalues. (An example is provided by $diag(1, \xi_1, \xi_1^2, \xi_2^2, \xi_2, \xi_1\xi_2^2, \xi_1, \xi_1^2\xi_2^2, \xi_2)$, $\xi_i^3 = 1$, which seems to have six different eigenvalues. Since there are only three different cubic roots of unity there are actually at most three different eigenvalues, arranged in several ways.)

First a word about notation. The zeroes are represented by dots, for better readability, ξ is a cubic root of unity in general, η is a cubic root $\neq 1$, (i.e. $\xi^3 = 1$, $\eta^2 = -1-\eta$), $\epsilon = \pm 1$, other Greek symbols are also roots of some polynomial equation given in the text. Symbols with different subscripts have independent values. Small latin letters $x, y, z, p, q, k, \ldots$ are free parameters. Capital letters have special properties, given in the text.

We would like to repeat again that the simple solutions of section 2 are not included. Also, solutions related by the transformations discussed are not mentioned separately, neither are those obtained from previous solutions by restricting parameters.

5.1. Nine different eigenvalues

First we have a solution with the full number of nine different diagonal elements:

$$R_{9.1} = \left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & p & \cdot & 1-q\eta^2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & p^2 & \cdot & (1-q\eta)x & \cdot & (1-q\eta)(1-q\eta^2) & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & qp^{-1}\eta^2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & q\eta & \cdot & -q(1-q\eta^2)/x & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & qp & \cdot & -\eta^2 q(1-q\eta) & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q^2p^{-2}\eta & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q^2p^{-1}\eta^2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q^2
\end{array}\right).$$

Here x is a free parameter that can be scaled to 1, since it must be non-zero. The $q = \eta p^2$ subcase of this solution was presented in [10].

5.2. *Eight different eigenvalues*

The first solution is the multiparameter version of the standard $GL(3)$ solution, as generalized in [11] and [12] to the following that has maximum of four free parameters, p, c, q, k:

$$R_{8.1} = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & p & \cdot & 1-q & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & c & \cdot & \cdot & \cdot & 1-q & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & qp^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & A & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & qk^{-1} & \cdot & 1-q & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & qc^{-1} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & k & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & B \end{array}\right).$$

where $A, B \in \{1, -q\}$.

If one imposes further relations among the diagonal entries there can be other non-zero matrix elements elsewhere. The first such solution is

$$R_{8.2} = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdot & \cdot & \cdot & \cdot & x & \cdot & -xp^2\mu^{-1} & \cdot \\ \cdot & p & \cdot & 1-\mu & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & p^{-1} & \cdot & \cdot & \cdot & 1-\mu & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & p^{-1}\mu & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & p^{-2}\mu^2 & \cdot & 1-\mu & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p\mu & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p^2\mu^{-1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^3 \end{array}\right)$$

where μ is a root of $(\mu^3 - 1)(\mu^2 + 1) = 0$. In fact $R_{8.2}$ has eight different eigenvalues when $\mu^2 = -1$ and 7 when $\mu^3 = 1$, but we keep these two branches together. If $x = 0$ we get a subcase of $R_{8.1}$. Other solutions of this type will appear later.

The following is a rather asymmetric solution:

$$R_{8.3} = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdot & \cdot & \cdot & \cdot & x & \cdot & \cdot & \cdot \\ \cdot & p & \cdot & 1-\xi & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & p^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & p^{-1}\xi & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & p^{-2}\xi^2 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p\xi^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p^2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \xi \end{array}\right).$$

5.3. Seven different eigenvalues

The following solution is related to $R_{9.1}$: if in the diagonal of $R_{9.1}$ the parameter p is restricted to be a cubic root of unity we can have more non-zero matrix elements:

$$R_{7.1} = \left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & \cdot & \cdot & \cdot & xy & \cdot & xy\xi^2(1-q\eta) & \cdot \\
\cdot & \xi & \cdot & 1-q\eta^2 & \cdot & \cdot & \cdot & \cdot & y\xi(1-q\eta^2) \\
\cdot & \cdot & \xi^2 & \cdot & (1-q\eta)x & \cdot & (1-q\eta)(1-q\eta^2) & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & q\xi^2\eta^2 & \cdot & \cdot & \cdot & \cdot & -yq\eta \\
\cdot & \cdot & \cdot & \cdot & q\eta & \cdot & -q(1-q\eta^2)/x & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & q\xi & \cdot & -\eta^2 q(1-q\eta) & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q^2\xi\eta & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q^2\xi^2\eta^2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q^2
\end{array}\right).$$

Here we have two scalable parameters x and y, and they can both be scaled to 1, because $x \neq 0$ and $y = 0$ only leads to a special case of $R_{9.1}$.

A restriction on the diagonal elements of $R_{8.1}$ allows

$$R_{7.2} = \left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & \cdot & \cdot & x & \cdot & \cdot & \cdot & \cdot \\
\cdot & \epsilon_1 & \cdot & 1-q & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & p & \cdot & \cdot & \cdot & 1-q & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & q\epsilon_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -q & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & p\epsilon_2 & \cdot & 1-q & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p^{-1}q & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p^{-1}q\epsilon_2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & A
\end{array}\right)$$

where $A \in \{1, -q\}$.

5.4. Six different eigenvalues

The following solution is related to $R_{9.1}$ and $R_{7.1}$

$$R_{6.1} = \left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & p & \cdot & 1-q^2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & p^2 & \cdot & (1-q^2)x & \cdot & (1+q)(1-q)^2 & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & q^2p^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & q & \cdot & q(1-q^2)/x & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & p & \cdot & (1-q^2) & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q^4p^{-2} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q^2p^{-1} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right).$$

The $p = -q$ subcase was found in [13].

The next solution is related to $R_{8.2}$: if we have $\mu = 1$ in $R_{8.2}$ there is other freedom available:

$$R_{6.2} = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdot & \cdot & \cdot & \cdot & x & \cdot & y & \cdot \\ \cdot & p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & p^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & p^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \epsilon & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & p^{-2}\epsilon & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p^2\epsilon & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \epsilon \end{array}\right).$$

Here x and y can be scaled to 1, if they are non-zero.

5.5. *Five different eigenvalues*

There were no new non-trivial solutions with five different eigenvalues.

5.6. *Four different eigenvalues*

First we have another relative of $R_{8.1}$

$$R_{4.1} = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdot & \cdot & \cdot & x & \cdot & \cdot & \cdot & \cdot \\ \cdot & \mu^2 & \cdot & 1-\epsilon\mathrm{i} & \cdot & \cdot & \cdot & \cdot & -\mathrm{i}\mu^2 y\epsilon \\ \cdot & \cdot & \mu & \cdot & \cdot & \cdot & 1-\epsilon\mathrm{i} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \mathrm{i}\mu^2\epsilon & \cdot & \cdot & \cdot & \cdot & z \\ \cdot & \cdot & \cdot & \cdot & -\mathrm{i}\epsilon & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \mu^3 & \cdot & 1-\epsilon\mathrm{i} & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathrm{i}\mu^3\epsilon & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathrm{i}\mu\epsilon & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\mathrm{i}\epsilon \end{array}\right)$$

where $\mu^4 = 1$.

Up to now the diagonal blocks have been of type C_1. The next two represent a pattern that we will meet several times later on; the diagonal and (1,3) blocks are of type C_3 (more or less) and in addition, there are some non-zero entries on the fourth off-diagonal.

$$R_{4.2} = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdot & x & \cdot & \cdot & \cdot & c & \cdot & a \\ \cdot & p & \cdot & \cdot & \cdot & q & \cdot & p(x+c-py-q) & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & c \\ \hline \cdot & \cdot & \cdot & p^{-1} & \cdot & y & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \epsilon & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & p^{-1} & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & x \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right)$$

$$R_{4.3} = \left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & x & \cdot & \cdot & \cdot & -x & \cdot & -xy \\
\cdot & p & \cdot & \cdot & \cdot & x-y & \cdot & -zp^2 & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -y \\
\hline
\cdot & \cdot & \cdot & p^{-1} & \cdot & z & \cdot & y-x & \cdot \\
\cdot & \cdot & \cdot & \cdot & \epsilon & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & p^{-1} & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & y \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right)$$

$$R_{4.4} = \left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & \cdot & \cdot & x & \cdot & y & \cdot & \cdot \\
\cdot & \epsilon_1 & \cdot & \cdot & \cdot & \cdot & \cdot & y\epsilon_1 & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & y \\
\hline
\cdot & \cdot & \cdot & \epsilon_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \epsilon_2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \epsilon_1 & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \epsilon_1 p & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p
\end{array}\right).$$

5.7. *Three different eigenvalues*

The next solution reduces to $R_{8.2}$ if $y = 0$, cf also $R_{7.1}$

$$R_{3.1} = \left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & \cdot & \cdot & \cdot & x & \cdot & -x\xi_1^2\xi_2^2 & \cdot \\
\cdot & \xi_1 & \cdot & 1-\xi_2 & \cdot & \cdot & \cdot & \cdot & -y\xi_1 \\
\cdot & \cdot & \xi_1^2 & \cdot & \cdot & \cdot & 1-\xi_2 & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \xi_1^2\xi_2 & \cdot & \cdot & \cdot & \cdot & y \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \xi_1\xi_2^2 & \cdot & 1-\xi_2 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \xi_1\xi_2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \xi_1^2\xi_2^2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right)$$

$$R_{3.2} = \left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & \cdot & \cdot & \cdot & x & \cdot & y & \cdot \\
\cdot & \xi & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p \\
\cdot & \cdot & \xi^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \xi^2 & \cdot & \cdot & \cdot & \cdot & xp\xi y^{-1} \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \xi & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \xi & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \xi^2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right)$$

$$R_{3.3}=\left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & \cdot & \cdot & \cdot & x & \cdot & x\xi_1^2(1-\xi_2) & \cdot \\
\cdot & \xi_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \xi_1^2 & \cdot & y & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \xi_1^2 & \cdot & \cdot & \cdot & \cdot & -\xi_2 x(1-\xi_2))y^{-1} \\
\cdot & \cdot & \cdot & \cdot & \xi_2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \xi_2^2\xi_1 & \cdot & -1+\xi_2 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \xi_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \xi_2^2\xi_1^2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \xi_2
\end{array}\right).$$

The following reduces to $R_{8.3}$ if $y=0$:

$$R_{3.4}=\left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & \cdot & \cdot & \cdot & x & \cdot & \cdot & \cdot \\
\cdot & \xi_1 & \cdot & 1-\xi_2\xi_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \xi_1^2 & \cdot & y & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \xi_2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \xi_2^2 & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \xi_2^2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \xi_1^2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \xi_2\xi_1
\end{array}\right).$$

5.8. Two different eigenvalues

Now most of the solution have the previously mentioned structure: the diagonal and (1,3) blocks are like C_3; in addition, there are some non-zero entries on the fourth off-diagonal. We list these solutions in decreasing number of non-zero elements on the fourth diagonal:

$$R_{2.1}=\left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & x & \cdot & p & \cdot & a & \cdot & q \\
\cdot & \epsilon_1 & \cdot & \cdot & \cdot & k & \cdot & a\epsilon_1+k\epsilon_1\frac{1}{2}(\epsilon_2-1) & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & a \\
\hline
\cdot & \cdot & \cdot & \epsilon_1 & \cdot & x\epsilon_1+k\epsilon_1\frac{1}{2}(\epsilon_2-1) & \cdot & k & \cdot \\
\cdot & \cdot & \cdot & \cdot & \epsilon_2 & \cdot & \cdot & \cdot & k^2p^{-1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \epsilon_1 & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & x \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \epsilon_1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right)$$

$$R_{2.2}=\left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & y-x & \cdot & p & \cdot & -y+x & \cdot & \frac{1}{2}(-2y^2+x^2) \\
\cdot & \epsilon & \cdot & \cdot & \cdot & -x & \cdot & -y\epsilon & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -(y+x) \\
\hline
\cdot & \cdot & \cdot & \epsilon & \cdot & y\epsilon & \cdot & x & \cdot \\
\cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & x^2p^{-1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \epsilon & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & y+x \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \epsilon & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right)$$

$$R_{2.3} = \left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & x & \cdot & a & \cdot & -x & \cdot & -xy \\
\cdot & \epsilon & \cdot & \cdot & \cdot & -y+x & \cdot & -y\epsilon & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -y \\
\hline
\cdot & \cdot & \cdot & \epsilon & \cdot & y\epsilon & \cdot & y-x & \cdot \\
\cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \epsilon & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & y \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \epsilon & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right)$$

$$R_{2.4} = \left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & x & \cdot & p & \cdot & a\epsilon & \cdot & q \\
\cdot & \epsilon & \cdot & \cdot & \cdot & x-k & \cdot & a & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & a\epsilon \\
\hline
\cdot & \cdot & \cdot & \epsilon & \cdot & k\epsilon & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \epsilon & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & x \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \epsilon & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right)$$

$$R_{2.5} = \left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & x & \cdot & p & \cdot & -x & \cdot & -xy \\
\cdot & \epsilon_1 & \cdot & \cdot & \cdot & \cdot & \cdot & -x\epsilon_1 & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -y \\
\hline
\cdot & \cdot & \cdot & \epsilon_1 & \cdot & x\epsilon_1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \epsilon_2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \epsilon_1 & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & y \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \epsilon_1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right).$$

For the following two the diagonal blocks are of type C_2. If the parameter b were zero one could make an index shuffle so that they would fit into the previously mentioned band form.

$$R_{2.6} = \left(\begin{array}{ccc|ccc|ccc}
1 & x & \cdot & -x & -x^2 & \cdot & \cdot & \cdot & a \\
\cdot & 1 & \cdot & \cdot & -x & \cdot & \cdot & \cdot & -b \\
\cdot & \cdot & \epsilon_1 & \cdot & \cdot & -x\epsilon_1 & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & 1 & x & \cdot & \cdot & \cdot & b \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \epsilon_1 & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \epsilon_1 & x\epsilon_1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \epsilon_1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \epsilon_2
\end{array}\right)$$

$$R_{2.7} = \left(\begin{array}{ccc|ccc|ccc}
1 & x & . & -x & x(x-2c) & . & . & . & -bx \\
. & 1 & . & . & x-2c & . & . & . & -b \\
. & . & \epsilon_1 & . & . & -c\epsilon_1 & . & . & . \\
\hline
. & . & . & 1 & -x+2c & . & . & . & b \\
. & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & \epsilon_1 & . & . & . \\
\hline
. & . & . & . & . & . & \epsilon_1 & c\epsilon_1 & . \\
. & . & . & . & . & . & . & \epsilon_1 & . \\
. & . & . & . & . & . & . & . & \epsilon_2
\end{array}\right).$$

The following are rather sparse but have more freedom on the diagonal

$$R_{2.8} = \left(\begin{array}{ccc|ccc|ccc}
1 & . & . & . & x & a & . & b & c \\
. & \epsilon_1 & . & . & . & . & . & . & . \\
. & . & \epsilon_1 & . & . & . & . & . & . \\
\hline
. & . & . & \epsilon_1 & . & . & . & . & . \\
. & . & . & . & \epsilon_2 & . & . & . & . \\
. & . & . & . & . & \epsilon_2 & . & . & . \\
\hline
. & . & . & . & . & . & \epsilon_1 & . & . \\
. & . & . & . & . & . & . & \epsilon_2 & . \\
. & . & . & . & . & . & . & . & \epsilon_2
\end{array}\right)$$

$$R_{2.9} = \left(\begin{array}{ccc|ccc|ccc}
1 & . & . & . & x & . & . & . & a \\
. & \epsilon_3 & . & . & . & . & . & . & . \\
. & . & \epsilon_1 & . & . & . & . & . & . \\
\hline
. & . & . & \epsilon_3 & . & . & . & . & . \\
. & . & . & . & \epsilon_4 & . & . & . & . \\
. & . & . & . & . & \epsilon_2 & . & . & . \\
\hline
. & . & . & . & . & . & \epsilon_1 & . & . \\
. & . & . & . & . & . & . & \epsilon_2 & . \\
. & . & . & . & . & . & . & . & \epsilon_5
\end{array}\right).$$

5.9. *One eigenvalue*

The next four solutions have basically the C_3 structure, with some extra elements.

$$R_{1.1} = \left(\begin{array}{ccc|ccc|ccc}
1 & . & x & . & . & . & -x & . & -xy \\
. & 1 & . & . & . & -y+x & . & -x & q \\
. & . & 1 & . & . & . & . & . & -y \\
\hline
. & . & . & 1 & . & x & . & y-x & -q \\
. & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . \\
\hline
. & . & . & . & . & . & 1 & . & y \\
. & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & 1
\end{array}\right)$$

$$R_{1,2} = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdot & x & \cdot & \cdot & \cdot & p & \cdot & a \\ \cdot & 1 & \cdot & \cdot & \cdot & p-q & \cdot & q & k \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & p \\ \hline \cdot & \cdot & \cdot & 1 & \cdot & x & \cdot & \cdot & -k \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & x \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right)$$

$$R_{1,3} = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdot & x & \cdot & \cdot & \cdot & -x & \cdot & -xy \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -y & -p \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -y \\ \hline \cdot & \cdot & \cdot & 1 & \cdot & y & \cdot & \cdot & p \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & q \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & y \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right)$$

$$R_{1,4} = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdot & x & \cdot & \cdot & \cdot & -x & \cdot & -xy \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -2y+x & p \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -y \\ \hline \cdot & \cdot & \cdot & 1 & \cdot & 2y-x & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & y \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right).$$

The next one is like C_2. The labeling change $2 \leftrightarrow 3$ would make it almost C_3 but would not keep the solution upper triangular.

$$R_{1,5} = \left(\begin{array}{ccc|ccc|ccc} 1 & x & \cdot & -x & -x^2 & -k+xq & p & k & a \\ \cdot & 1 & \cdot & \cdot & -x & \cdot & \cdot & q & b \\ \cdot & \cdot & 1 & \cdot & \cdot & -x & \cdot & \cdot & p \\ \hline \cdot & \cdot & \cdot & 1 & x & \cdot & \cdot & p-q & -b \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & x & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right).$$

The next three solutions have mixed diagonal blocks. It could be possible to transform the upper block to C_2 or C_3 but we chose a different and apparently simpler form defined by the requirement that the above-mentioned relabeling $2 \leftrightarrow 3$ keeps the matrix upper triangular.

$$R_{1.6}=\left(\begin{array}{ccc|ccc|ccc}
1 & x & p & -x & -xy & -2px-k & -p & k & -pq\\
. & 1 & . & . & -y & . & . & -p & .\\
. & . & 1 & . & . & -x & . & . & -q\\
\hline
. & . & . & 1 & y & p & . & . & .\\
. & . & . & . & 1 & . & . & . & .\\
. & . & . & . & . & 1 & . & . & .\\
\hline
. & . & . & . & . & . & 1 & x & q\\
. & . & . & . & . & . & . & 1 & .\\
. & . & . & . & . & . & . & . & 1
\end{array}\right)$$

$$R_{1.7}=\left(\begin{array}{ccc|ccc|ccc}
1 & x & p & -x & -xy & c & a & -c+x(-q+a) & b\\
. & 1 & . & . & -y & -q+p & . & a & .\\
. & . & 1 & . & . & -x & . & . & a\\
\hline
. & . & . & 1 & y & q & . & . & .\\
. & . & . & . & 1 & . & . & . & .\\
. & . & . & . & . & 1 & . & . & .\\
\hline
. & . & . & . & . & . & 1 & x & p\\
. & . & . & . & . & . & . & 1 & .\\
. & . & . & . & . & . & . & . & 1
\end{array}\right)$$

$$R_{1.8}=\left(\begin{array}{ccc|ccc|ccc}
1 & x & q & -x & -x^2 & -k & -q & k & px\\
. & 1 & . & . & -x & q & . & . & p\\
. & . & 1 & . & . & -x & . & . & .\\
\hline
. & . & . & 1 & x & . & . & -q & -p\\
. & . & . & . & 1 & . & . & . & .\\
. & . & . & . & . & 1 & . & . & .\\
\hline
. & . & . & . & . & . & 1 & x & .\\
. & . & . & . & . & . & . & 1 & .\\
. & . & . & . & . & . & . & . & 1
\end{array}\right).$$

Finally we have two solutions whose diagonal blocks are of type C_5:

$$R_{1.9}=\left(\begin{array}{ccc|ccc|ccc}
1 & 1 & . & -1 & -1 & x & y & y-x & p\\
. & 1 & 1 & . & -1 & -1 & . & y & -z+y\\
. & . & 1 & . & . & -1 & . & . & y\\
\hline
. & . & . & 1 & 1 & . & -1 & -1 & z\\
. & . & . & . & 1 & 1 & . & -1 & -1\\
. & . & . & . & . & 1 & . & . & -1\\
\hline
. & . & . & . & . & . & 1 & 1 & .\\
. & . & . & . & . & . & . & 1 & 1\\
. & . & . & . & . & . & . & . & 1
\end{array}\right).$$

In the next one the parametrization is rather complicated and it would be interesting to understand its origin. (Note that $x=1$ yields a symmetric solution.)

$R_{1.10}$

$$
= \left(\begin{array}{ccc|ccc|ccc}
1 & 2 & . & -2 & -4x & 8x & 4 & . & -4x(7+x) \\
. & 1 & 2 & . & -2x & 4-6x-2x^2 & . & 2x(1+x) & 6x(2+x)(-1+x) \\
. & . & 1 & . & . & 2-4x & . & . & -4x(1-2x) \\
\hline
. & . & . & 1 & 2x & -2x(1-x) & -2 & -2x(1+x) & -2x(1+3x)(-2+x) \\
. & . & . & . & 1 & 2x & . & -2x & 4x(1-2x) \\
. & . & . & . & . & 1 & . & . & 2-4x \\
\hline
. & . & . & . & . & . & 1 & -2+4x & 4(1-2x)(1-x) \\
. & . & . & . & . & . & . & 1 & -2+4x \\
. & . & . & . & . & . & . & . & 1
\end{array}\right).
$$

6. Discussion

In this paper we have given all upper triangular non-singular solutions to the constant quantum Yang–Baxter equation (1), modulo upper triangular transformations (13) and reflections (14), (15) and minus simple solutions of section 2.

There is not much that we can say now about the solutions; here are only some random observations. (i) Some of the solutions ($R_{9.1}$, $R_{8.1}-R_{8.3}$, $R_{7.1}$, $R_{6.1}$, $R_{6.2}$, $R_{4.1}$ and $R_{3.1}-R_{3.4}$) satisfy also the weight condition $R_{ij}^{kl}=0$, if $k+l \neq i+j \pmod 3$. Several others satisfy the same condition $\pmod 2$, e.g. $R_{2.1}-R_{2.5}$. (ii) The upper triangular property is of course sensitive to index relabeling, for example $2 \leftrightarrow 3$ often breaks it. Sometimes one can find an upper triangular transformation after which the solutions stays upper triangular even after $2 \leftrightarrow 3$ exchange. This happens for the solutions $R_{8.3}$, $R_{4.2}$, $R_{4.4}$, $R_{3.4}$, $R_{2.4}$, $R_{2.5}$, $R_{2.9}$, $R_{1.4}$, $R_{1.6}$, $R_{1.7}$ and $R_{1.8}$, and they are presented in this special form. (iii) Some solutions show interesting parametric relations, in particular $R_{9.1}$, $R_{7.1}$, $R_{6.1}$ and $R_{1.10}$.

Now that we have these constant solutions several natural questions come up for further study, for example, can one add a spectral parameter, can one contruct a corresponding universal R-matrix? The corresponding algebraic structures also need investigation.

Appendix A.

In this appendix we give the breakdown of the original problem to smaller subsets. The primary classification proceeds by the upper triangular blocks on the diagonal. The 'number of solutions' given below is the number before any trivial solutions or subcases were eliminated.

A. Upper block of type C_5

We use scaling freedom to put all non-zero entires in C_5 to 1. The solution procedure produced 8 solutions, which were immediately reduced to 3 basic solutions.

B. Upper block of type C_4

Let us scale so that $b=c=1$ in C_4. In principle we could exclude lower blocks of type C_5 from the very beginning, but no such solutions were found anyway.

(i) a arbitrary, $c_5 \neq 0$ or $f_7 \neq 0$: 3 solutions, all have $a=1$.
(ii) $a \neq 1$, $c_5 = f_7 = 0$: 22 solutions

(iii) $a = 1$, $c_5 = f_7 = 0$: 1) $b_7 \neq 0$, 4 solutions. 2) $b_7 = 0$, $a_8 = 1$, 9 solutions. 3) $b_7 = 0$, $a_8 = 0$: 35 solutions.

C. Upper block of type C_3
We can scale $a = b = 1$.

C.1. $c_5 \neq 0$ or $f_7 \neq 0$
no solutions.

C.2. $c_5 = f_7 = 0$, $c \neq 1$
33 solutions.

C.3. $c_5 = f_7 = 0$, $c = 1$

Now C_3 commutes with any upper triangular matrix with units on the diagonal. Using this we can put the lower block also into a semicanonical form. Solving some of the equations reveals that all the diagonal entries of the lower block also equal 1. If the lower block is of type C_5 a reflection takes the system to the one studied before; the same holds if the lower block is of type C_2. If the lower block is of type C_4, we get a single solution.

What remains is a lower block of type C_3 with $a = c$, but possibly with $b = 0$. This again commutes with the UT transformation matrix with units on the diagonal; we will thus use this rotational freedom to put the centre block in a semicanonical form.

(i) Centre block of type 5: no solutions.
(ii) Centre block of type 2: 3 solutions.
(iii) Centre block of type 3: 42 solutions.

D. Upper block of type C_2
Now the lower block must be of type C_4 or C_1, all others can be reflected to the cases studied before.

(i) $f_7 \neq 0$: No solutions.
(ii) $c_5 \neq 0$ or $f_7 = 0$: 2 solutions.
(iii) $c_3 \neq 1$, $c_5 = f_7 = 0$: 11 solutions
(iv) $c_3 = 1$, $c_5 = f_7 = 0$: In this case one quickly finds that the lower block must be

$$\begin{pmatrix} k & h & g \\ . & k & . \\ . & . & l \end{pmatrix}.$$

Here we need to analyse only the case $h = g = 0$ because (1) If $k = l$, $h \neq 0$ we can use upper triangular transformations to put $g = 0$. Then the lower block is of type C_2, which reflects to an upper block of type C_4 done earlier. (2) If $k = l$, $h = 0$, $g \neq 0$ lower block is of type C_3, done earlier. (3) If $k \neq l$, $h \neq 0$ we can transform lower block to type C_2 without changing the upper block. (4) In the remaining cases the lower block is diagonal; there were 15 solutions of this type.

E. Upper block is non-unit diagonal, $c_5 \neq 0$, $f_7 \neq 0$
Let us normalize the diagonal elements as $(1, b, c)$. Here we may also assume that the lower block is diagonalizable by an upper triangular matrix, but maybe not yet in diagonal form. In most cases solving the first few equations yields directly a diagonal lower block, exceptions to this are discussed separately.

In general one finds that if $c_5 \neq 0$ or $f_7 \neq 0$ then $c_3 = b_2^2$, this implies in particular that in this section we must have $b_2 \neq 1$. After that one also finds quickly that the lower block must be

$$\begin{pmatrix} h & . & g \\ . & k & . \\ . & . & l \end{pmatrix} \tag{A1}$$

(i) $b^3 \neq 1, b^2 \neq 1$: 3 solutions.
(ii) $b = -1$: As mentioned above we must now have $c_3 = 1$. If now $l = h$ in (A1) the system can be reflected to one of the cases studied before. If $l \neq h$ one can use the transformation

$$\begin{pmatrix} 1 & . & y \\ . & 1 & . \\ . & . & 1 \end{pmatrix}$$

which commutes with $diag(1, -1, 1)$, to put $g = 0$. One finds 4 solutions.
(iii) $b^3 = 1$, $b \neq 1$: 5 solutions.

F. Upper block is non-unit diagonal, $c_5 \neq 0$, $f_7 = 0$
The same comments as above holds in this case and we just give the results.

(i) $b^3 \neq 1, b^2 \neq 1$: 4 solutions.
(ii) $b = -1$: 4 solutions.
(iii) $b^3 = 1$, $b \neq 1$: 6 solutions.

G. Upper block is non-unit diagonal, $c_5 = 0$, $f_7 \neq 0$
As before, the lower block must be as in (A1). If $l = h$ we can reflect the system to case F or C, else we eliminate g and reflect to case F.

H. Upper block is non-unit diagonal, $c_5 = 0$, $f_7 = 0$

H.1. The diagonal elements $(1, b, c)$ *are all different*
76 solutions

H.2. $c = 1$, $b \neq 1$
Again one finds that the lower block must be of type (A1) and as in E(ii) one can show that the lower block must in fact be diagonal. If its diagonal elements are all different we reflect it to H.1. Thus we get subcases according to the diagonal entries of the lower block:

(i) Lower block $= diag(l, k, l)$: 21 solutions
(ii) Lower block $= diag(k, k, l)$ or $diag(k, l, l)$: 37 solutions.
(iii) Lower block $= diag(l, l, l)$: 19 solutions.

H.3. $b = 1$, $c \neq 1$
Now we may assume that the lower block is transformable to a diagonal form, which has at most two different elements and is not (h, l, h). All other cases can be reflected to ones studied before. Then one finds that the lower block must be

$$\begin{pmatrix} h & g & . \\ . & k & . \\ . & . & l \end{pmatrix}. \tag{A2}$$

Since the upper block stays invariant under transformations

$$\begin{pmatrix} 1 & y & . \\ . & 1 & . \\ . & . & 1 \end{pmatrix}$$

we can transform the case $h \neq k$ to a diagonal form while the case $h = l$ reflects to C_4.

(i) Lower block is $diag(h, h, l)$: 36 solutions.
(ii) Lower block is $diag(h, l, l)$: 18 solutions.
(iii) Lower block is $diag(l, l, l)$: 36 solutions.

H.4. $b = c \neq 1$
Again one can trasform the lower block to something that reflects to one of the cases before, except for two cases:

(i) Lower block is $diag(h, h, l)$: 17 solutions.
(ii) Lower block is $diag(l, l, l)$: 40 solutions

I. Upper and lower blocks are unit matrices
In this case we have the full trasformation freedom left and we use it to bring the centre block into the semicanonical form (recall C_2 reflects to C_4)

(i) Centre block type C_5: no solutions.
(ii) Centre block type C_2: 2 solutions.
(iii) Centre block type C_3: 13 solutions.
(iv) Centre block type C_1: Let us call the diagonal elements of the centre block as (d, f, g). (1) d, f, g all different, 4 solutions. (2) $g = d$, $f \neq d$, 6 solutions. (3) $d = f$, $f \neq g$, 9 solutions. (4) Centre block unit matrix, 9 solutions.

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